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A generalized parametric equation of state describing the global behaviour of an Ising-like system

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Abstract. We report a microscopic derivation of the parametric representation of the equation of state for an Ising-like system by means of the skeleton expansion technique. Such a representation explicitly describes the crossover behaviour from the small correlation length region to the critical region. The Gaussian–Ising crossover is obtained at criticality. The derivation is based on differential relations that connect natural thermodynamical variables with the parameters defining the representation. Differentiability properties are checked in the small ϵ limit.

1. Introduction

The parametric representation of the equation of state has at least two main advantages. The first of these is to verify the Griffiths analyticity requirements (Griffiths 1967); the second is to introduce new thermodynamic variables θ and R associated respectively with the regular and singular behaviour around the critical point. The simplest version of the parametric equation of state, the 'linear model' (Shofield 1969a, b, Ho and Litster 1969), has been criticized from many points of view.

A theoretical check of such a model has been performed by several methods. It has been shown by series expansion (Gaunt and Domb 1970) that the model is not correct in two and three dimensions even for Ising systems. By ϵ expansion ($\epsilon = 4-d$ where d is the space dimensionality), deviations from the linear model have been calculated to the order of ϵ^2 (Brézin *et al* 1973); in particular its failure for Ising systems has been demonstrated up to $O(\epsilon^3)$ (Wallace and Zia 1974). In addition the extension for an isotropic system with $n \geqslant 2$ (where n is the number of internal degrees of freedom of the spin density) is obscure because of the expected singularity near the coexistence curve. Several inadequacies have been expressed even in the phenomenological analysis (Barmatz *et al* 1975) and a possible generalization of the parametric formulation of scaling has been proposed which, besides suggesting possible forms for correction terms, gives the possibility of describing the non-rectilinear behaviour of the diameter of the coexistence curve (Green *et al* 1971).

All this might be clarified by introducing the parametric representation by a direct microscopic approach which gives the linear model as a first step of a systematic approximation scheme. This is the first aim of our paper.

The results obtained to the leading order in ϵ will be identified in the asymptotic critical region with the most recent derivation of the equation of state based on the differential renormalization group (Rudnick and Nelson 1976, Nelson 1976). These results are an improvement of the first 'naive' derivation (Brézin *et al* 1973, Avdeeva and Migdal 1972, Brézin *et al* 1976) because of the explicit agreement with the Griffiths conditions for large x (where $x = t/\phi^{1/\beta}$, t is the reduced temperature $t = (T - T_c)/T_c$, ϕ is the average value of the order parameter and β is the critical exponent describing the coexistive curve). Our approach actually gives a generalization of the equation of state valid even outside the critical region. As far as we consider the dependence on the reduced temperature t, the magnetic field h and the strength of the quartic coupling g between fluctuations, in various asymptotic regimes, we recover in addition to the critical behaviour the tricritical-like and the small correlation length regimes. The results in the zero magnetic field case are in agreement, to the leading order in ϵ, with previous works on the crossover between critical and tricritical-like behaviour (Rudnick and Nelson 1976, Lawrie 1976, Bruce and Wallace 1976).

The first problem we will deal with is a methodological matter: how to define on microscopic grounds the parametric representation of 'natural' variables, i.e. the average value of the order parameter ϕ and the temperature t, in terms of 'intrinsic' parameters R and θ, with a fixed bare coupling constant g.

In a previous work (Bouché *et al* 1975) it has been shown that, in the presence of an external field, a suitable version of the skeleton expansion technique (de Pasquale and Tombesi 1972, Tsuneto and Abrahams 1973, Ginzburg 1974) allows the introduction of differential relations between the natural thermodynamic variables and two effective coupling constants u and v associated with the structure of the skeleton diagrams. In that work it has been shown that such a set of differential relations, when integrated along particular paths, generates asymptotic solutions near the critical point that look like the linear model approximation of the parametric equation of state and, to the first order in ϵ, are in agreement with Brézin *et al* (1973).

In the present work we shall investigate the general solution of the same set of equations which is consistent with a prescribed behaviour near the infinite Gaussian fixed point (Nicoll *et al* 1976) ($u = 0$, $v = 0$, g fixed), i.e. the asymptotic region of vanishingly small correlation length. The point is to check the differentiability of the natural thermodynamic variables with respect to the intrinsic variables u and v. Such a differentiability can be verified in the whole range, between the small correlation length regime and the critical region, in the context of a small ϵ limit. We emphasize that in the present work the ϵ limit concept turns out to be useful in order to check a qualitative aspect of the theory. It is straightforward to verify that the one-loop skeleton model (Bouché *et al* 1975) verifies the differentiability property only to the leading order in ϵ. Further steps in the skeleton expansion would lead to verification of differentiability to orders next to the leading one. We shall confine ourselves to a discussion of the results of the first step. The solution we obtain will depend on the bare coupling constant g which comes out as a boundary condition on the differential equation near the infinite Gaussian fixed point.

To discuss the *tricritical* point we must allow both t and g to be small independently. We will see that for vanishingly small g our generalized parametric equation will predict a finite Gaussian fixed point ($u = 0$, $v = 0$ and $g = 0$) which is believed to govern the tricritical phenomena. As a consequence we will be able to discuss the crossover between the critical and the tricritical behaviour in the presence of the field h.

2. The model

We introduce a slightly modified version of the usual Landau–Ginzburg–Wilson (Wilson and Kogut 1974) model for the Ising system. We find it convenient to introduce, by a dummy functional integration (Halperin *et al* 1974, Coleman *et al* 1974), a new field σ which describes weak fluctuations, i.e. energy fluctuations, of the system. Such a field is coupled with an external field J that has the physical meaning of temperature. The strong fluctuations, i.e. the order parameter fluctuations, are described by the field ψ coupled with the external magnetic field h. The partition functional is

$$Z\{h, J\} = \int \delta\psi \int \delta\sigma \exp\Big(- \int \mathrm{d}^d x [\tfrac{1}{2}(\nabla\psi(x))^2 + \tfrac{1}{2}\mathrm{i}g\sigma(x)\psi^2(x) + \tfrac{1}{2}\sigma^2(x) - h(x)\psi(x) - J(x)\sigma(x)]\Big). \tag{2.1}$$

The one-particle irreducible Green functions or vertices, are associated with the functional $\Gamma\{\phi, J\}$ obtained from the partition functional by means of a Legendre transformation

$$\Gamma\{\phi, J\} = \ln Z\{h, J\} - \int \mathrm{d}^d x\, h(x)\phi(x) \tag{2.2}$$

where $\phi(x)$ is the average value of the field $\psi(x)$ defined also as

$$\phi(x) = (\delta \ln Z\{h, J\}/\delta h(x))_J. \tag{2.3}$$

The vertices are defined as

$$\Gamma_{n,m}\{\phi, J\} = \delta^{n+m}\Gamma\{\phi, J\}/\delta\phi^n\, \delta J^m. \tag{2.4}$$

The one-loop skeleton model is extensively discussed by Bouché *et al* (1975), so we report here only the basic definitions and the equations of interest.

The dimensionless coupling constants u and v are defined by

$$u = \Gamma_{4,0}(\Gamma_{2,0})^{-\epsilon/2} i_3; \qquad v = (\Gamma_{3,0})^2(\Gamma_{2,0})^{-[1+(\epsilon/2)]} i_3 \tag{2.5}$$

where $\Gamma_{4,0}$ and $\Gamma_{3,0}$ are respectively the Fourier transforms of the four- and three-leg vertices calculated at zero external momenta, $\Gamma_{2,0}$ is the Fourier transform of the inverse of the order parameter–order parameter correlation function at vanishing external momenta and $i_3 = (2\pi)^{-d} \int \mathrm{d}^d y (1+y^2)^{-3}$. The equations defining the model are those following (Bouché *et al* 1975):

$$\delta u = \Big[15u^2 - 10uv\Big(1+\frac{\epsilon}{2}\Big) - \frac{\epsilon}{2}u + \Big(1+\frac{\epsilon}{2}\Big)\Big(2+\frac{\epsilon}{2}\Big)v^2\Big]\delta_\phi \ln \Gamma_{2,0} + \Big[\Big(1+\frac{\epsilon}{2}\Big)\Big(2+\frac{\epsilon}{2}\Big)v^2 - 6uv\Big(1+\frac{\epsilon}{2}\Big) + 3u^2 - \frac{\epsilon}{2}u\Big]\delta_J \ln \Gamma_{2,0} \tag{2.6a}$$

$$\delta v = \Big[2u - \Big(1+\frac{\epsilon}{2}\Big)v\Big]\delta_\phi \ln \Gamma_{2,0} + \Big[6uv - 2\Big(1+\frac{\epsilon}{2}\Big)v^2 - \Big(1+\frac{\epsilon}{2}\Big)v\Big]\delta_J \ln \Gamma_{2,0} \tag{2.6b}$$

$$\delta_\phi \ln \Gamma_{2,0} = (\Gamma_{3,0}/\Gamma_{2,0})\, \delta\phi; \qquad \delta_J \ln \Gamma_{2,0} = (\Gamma_{2,1}/\Gamma_{2,0})\, \delta J \tag{2.6c}$$

$$\delta \ln \Gamma_{2,1} = \Big[3u - \Big(1+\frac{\epsilon}{2}\Big)v\Big]\delta_\phi \ln \Gamma_{2,0} + \Big[u - \Big(1+\frac{\epsilon}{2}\Big)v\Big]\delta_J \ln \Gamma_{2,0} \tag{2.6d}$$

$$\delta\Gamma_{1,1}=\Gamma_{2,1}\,\delta\phi+i_3\Gamma_{2,1}\Gamma_{3,0}(\Gamma_{2,0})^{-\epsilon/2}\,\delta_J\ln\Gamma_{2,0} \tag{2.6e}$$

$$\delta h\equiv\delta\Gamma_{1,0}=\Gamma_{2,0}\,\delta\phi+\Gamma_{1,1}\,\delta J \tag{2.6f}$$

$$\delta\langle\sigma\rangle\equiv\delta\Gamma_{0,1}=\Gamma_{1,1}\,\delta\phi+\Gamma_{0,2}\delta J \tag{2.6g}$$

$$\delta\Gamma_{2,0}=\Gamma_{3,0}\,\delta\phi+\Gamma_{2,1}\,\delta J \tag{2.6h}$$

$$\delta\Gamma_{0,2}=(\Gamma_{2,1})^2 i_3(\Gamma_{2,0})^{-\epsilon/2}\delta\ln\Gamma_{2,0}. \tag{2.6i}$$

All vertices $\Gamma_{n,m}$ involved in equations (2.6) are actually the Fourier transforms at zero external momenta of the vertices defined in equation (2.4).

A peculiar feature of equations (2.6) is the role played by the effective coupling constants u and v. It is indeed possible to invert such equations obtaining differential equations that relate the vertices $\Gamma_{n,m}$, the average value of the order parameter ϕ and the temperature field J to the effective coupling constants u and v. Such differential relations locally define a parametric representation of the natural thermodynamic variables ϕ and J, associated with the original ensemble, in terms of the new set of variables u and v. Such a representation is meaningful only if the differentiability of the original variables with respect to u and v is verified. Equations (2.6) verify the differentiability requirements in the neighborhood of the origin of the (u, v) plane to the leading order and to the next to leading order in u and v and, in any finite region, to the leading order in ϵ.

Once the application of the differentiability criterion has selected the consistent form of differential equations of various quantities of interest, the integration procedure introduces an automatic summing up with respect to ϵ.

The solution is more conveniently expressed in terms of the variables† $u'=u/(\epsilon/6)$, $\theta^2=v/3u$, $\phi'=\mathrm{i}g\phi$, $t'=\mathrm{i}g(J-J_c)$, $h'=-\mathrm{i}gh$ and $\langle\sigma\rangle'=-\mathrm{i}g\langle\sigma\rangle$ (for simplicity we drop the prime from now on):

$$\begin{aligned}\phi&=G^{1/\epsilon}\theta(1-u)^{(1/\epsilon)-(1/2)}u^{-1/\epsilon}; \qquad G=3i_3g^2\\ t&=G^{2/\epsilon}(1-\tfrac{3}{2}\theta^2)(1-u)^{(2/\epsilon)-(1/3)}u^{-2/\epsilon}\end{aligned} \tag{2.7a}$$

$$\begin{aligned}h&=G^{3/\epsilon}\theta(1-\theta^2)(1-u)^{(3/\epsilon)-(1/2)}u^{-3/\epsilon}\\ \langle\sigma\rangle&=(G^{2/\epsilon}/C)\{(1-\tfrac{3}{2}\theta^2)[C-(2/\epsilon)+(2/\epsilon)(1-u)^{-1/3}]+\tfrac{1}{2}\theta^2(1-u)^{-1/3}\}\\ &\quad\times(1-u)^{(2/\epsilon)-(1/3)}u^{-2/\epsilon}\end{aligned} \tag{2.7b}$$

$$\Gamma_{2,0}=G^{2/\epsilon}[(1-u)/u]^{2/\epsilon} \tag{2.7c}$$

$$\Gamma_{0,2}=C-(2/\epsilon)+(2/\epsilon)(1-u)^{-1/3} \tag{2.7d}$$

where $C=\Gamma_{0,2}(0)$ is the limiting value of the specific heat as $t\to\infty$.

3. Crossover from small correlation length behaviour to criticality

Equations (2.7) define a parametric representation where the dependence on the anomaly θ is the same as predicted by the linear model in the zero ϵ limit‡ and the singular behaviour is included in the dependence on the variable u. We see that for arbitrary θ and fixed G, as u varies from zero to unity, the inverse susceptibility $\Gamma_{2,0}$

† The variable t' denotes the reduced temperature. This is easily seen by identifying our partition functional equation (2.1) with the usual one through a functional integration on the field σ.

‡ The zero ϵ limit for the linear model can be easily performed by using the ϵ expansion for the exponents.

varies from infinity to zero and, correspondingly, the reduced temperature varies from infinity to zero. As a consequence we have a description of the asymptotic critical region ($u \simeq 1$), the small correlation length region ($u \simeq 0$) and the crossover region. In the two previous limits the variable R of the usual linear model can be identified with $(1-u)^{(2/\epsilon)-(1/3)}$ and $u^{-2/\epsilon}$ respectively. In the same limits it is also possible to derive the standard expression for the equation of state,

$$h = \phi^{\delta} h(x) \qquad \text{where } x = t/\phi^{1/\beta} \tag{3.1}$$

$$h(x) = (x + \tfrac{1}{2}) \qquad \beta = \tfrac{1}{2}, \qquad \delta = 3 \tag{3.2}$$

which is the classical equation of state. In the $u \simeq 1$ limit we have the coupled equations

$$\begin{gathered} h(x) = G^{-2/(2-\epsilon)}\theta(x)^{-4/(2-\epsilon)}(1-\theta^2(x)) \\ x = G^{-(2/3)[2/(2-\epsilon)]}\frac{1-\frac{3}{2}\theta^2}{\theta^{[2-(\epsilon/3)]/[1-(\epsilon/2)]}}, \qquad \beta = \frac{1-(\epsilon/2)}{2-(\epsilon/3)}; \qquad \delta = \frac{3-(\epsilon/2)}{1-(\epsilon/2)}. \end{gathered} \tag{3.3}$$

By a suitable rescaling of h, θ, and x it is straightforward to identify equations (3.3) with equations (3.18) and (3.23) of Nelson (1976).

In the crossover region between $u = 0$ and $u = 1$ it is of physical interest to define effective critical exponents (Riedel and Wegner 1974), along the critical isotherm ($\theta^2 = 2/3$) and the coexistence curve ($\theta^2 = 1$).

We have

$$\left(\frac{\mathrm{d}\ln\phi}{\mathrm{d}\ln t}\right)_{\theta^2=1} \equiv \beta_{\text{eff}} = \frac{2-(\epsilon/3)u_{\text{CX}}(t)}{1-(\epsilon/2)u_{\text{CX}}(t)}; \qquad u_{\text{CX}}(t) \simeq \left(1 + 2^{\epsilon/2}\frac{|t|^{\epsilon/2}}{G}\right)^{-1} \tag{3.4a}$$

$$\left(\frac{\mathrm{d}\ln h}{\mathrm{d}\ln\phi}\right)_{\theta^2=2/3} \equiv \delta_{\text{eff}} = \frac{3-(\epsilon/2)u_{\text{CI}}(h)}{1-(\epsilon/2)u_{\text{CI}}(h)}; \qquad u_{\text{CI}}(h) \simeq \left\{1 + \left[\frac{1}{3}\left(\frac{2}{3}\right)^{1/2}\right]^{-\epsilon/3}\frac{h^{\epsilon/3}}{G}\right\}^{-1} \tag{3.4b}$$

where the dependence of u on t and h is given in the small ϵ limit.

As far as the susceptibility is concerned, i.e. $\Gamma_{2,0}^{-1}$, we obtain

$$\left(\frac{\mathrm{d}\ln\Gamma_{2,0}}{\mathrm{d}\ln t}\right)_{\theta^2=1} \equiv \gamma_{\text{eff}} = \frac{1}{1-(\epsilon/6)u_{\text{CX}}(t)} \tag{3.5a}$$

$$\left(\frac{\mathrm{d}\ln\Gamma_{2,0}}{\mathrm{d}\ln h}\right)_{\theta^2=2/3} = \frac{2}{3}\,\frac{1}{1-(\epsilon/6)u_{\text{CI}}(h)} = \frac{\delta_{\text{eff}}-1}{\delta_{\text{eff}}}. \tag{3.5b}$$

In the limit of vanishingly small external field, i.e. $\theta = 0$, we obtain for γ_{eff} the same result as in de Pasquale *et al* (1976).

The specific heat at constant magnetization is assumed to be proportional to the weak fluctuations correlation function $\Gamma_{0,2}$. The singular part of the specific heat can be defined by subtracting its limiting value in the asymptotic region of small correlation length. As a consequence we define the effective critical exponent α_{eff}:

$$\alpha_{\text{eff}} = -\left(\frac{\mathrm{d}\ln(\Gamma_{0,2}(u)-\Gamma_{0,2}(0))}{\mathrm{d}\ln t}\right)_{\theta^2=1} = \frac{\epsilon u_{\text{CX}}(t)}{(6-\epsilon u_{\text{CX}}(t)[1-(1-u_{\text{CX}}(t))^{-1/3}]}. \tag{3.6}$$

Along the critical isotherm we have

$$-\left(\frac{\mathrm{d}\ln(\Gamma_{0,2}(u)-\Gamma_{0,2}(0))}{\mathrm{d}\ln h}\right)_{\theta^2=2/3} = \frac{1}{3}\,\frac{\epsilon u_{\text{CI}}(h)}{[3-(\epsilon/2)u_{\text{CI}}(h)][1-(1-u_{\text{CI}}(h))^{-1/3}]}. \tag{3.7}$$

We note that all the scaling laws are verified in the two asymptotic regimes.

As a general comment we note that the amplitude of the crossover region is governed by the strength of the interaction G. We have the classical behaviour for smaller and smaller t as G decreases. In the next section we shall see that $G=0$ can be associated with the tricritical point. As a consequence this phenomenon can be considered as the competition between the tricritical and the critical behaviour in the pre-asymptotic region.

As far as the average value of the energy density field σ is concerned, we note a crossover phenomenon along the coexistence curve from a rectilinear behaviour far from the critical point to an enhanced curvature region close to it:

$$\langle\sigma\rangle_{\theta^2=1}=|t|\left[A+B\left(\frac{2^{\epsilon/2}(|t|^{\epsilon/2}/G)}{1+2^{\epsilon/2}(|t|^{\epsilon/2}/G)}\right)^{-1/3}\right]; \qquad A=\frac{2}{\epsilon C}-1 \qquad B=\frac{1}{C}-\frac{2}{\epsilon C}. \tag{3.8}$$

Near the critical point we have

$$\langle\sigma\rangle_{\theta^2=1}\simeq\left(\frac{2}{\epsilon C}-1\right)|t|+\left(\frac{1}{C}-\frac{2}{\epsilon C}\right)\frac{2^{-\epsilon/6}}{G^{-1/3}}|t|^{1-\alpha} \tag{3.9}$$

where $\alpha=\epsilon/6$. Such behaviour for the rectilinear diameter has been proposed by Ley-Koo and Green (1976) as a manifestation of a crossover to mean-field-like behaviour.

4. Gaussian–Ising crossover

Although the analysis thus far has been concentrated on critical behaviour and on the crossover from criticality to the small correlation length region, it is straightforward to treat the crossover from *tricritical* behaviour as well. To discuss the tricritical point we must allow both u and G to tend to zero independently at constant θ.

First we note from equation (2.7c) that the critical line is defined by

$$K-G=0 \tag{4.1}$$

where $K=G/u$.

We have only two asymptotic paths approaching the point $K=G=0$ of the (G, K) plane which are of physical interest. Any path asymptotically tangent to the $K=G$ line leads to the ordinary critical behaviour; any path asymptotically orthogonal to the G axis leads to tricritical behaviour. Other paths turn out to describe a trivial free-field theory.

Near the tricritical point it is useful to introduce the new variables

$$T=\frac{t}{G^{2/\epsilon}}(1-u)^{1/3}, \qquad \Phi=\frac{\phi}{G^{1/\epsilon}}(1-u)^{1/2}, \qquad H=\frac{h}{G^{3/\epsilon}}(1-u)^{1/2}. \tag{4.2}$$

The tricritical point is associated with infinitely large T, Φ and H. We obtain the parametric equations:

$$\Phi=\theta[(1-u)/u]^{1/\epsilon} \tag{4.3}$$

$$T=(1-\tfrac{3}{2}\theta^2)[(1-u)/u]^{2/\epsilon} \tag{4.4}$$

$$H=\theta(1-\theta^2)[(1-u)/u]^{3/\epsilon}. \tag{4.5}$$

In terms of these new variables we have the 'classical' equation of state

$$H/\Phi^3 = X + \tfrac{1}{2}; \qquad X = T/\Phi^2. \tag{4.6}$$

From equations (4.3) and (4.4) we have

$$u = \frac{1}{1 + (T + \frac{3}{2}\Phi^2)^{\epsilon/2}} \tag{4.7}$$

which together with equation (4.2) can be solved iteratively in the neighbourhood of $u = 0$. The explicit expression of the equation of state (4.6) in terms of the physical variables h, t, ϕ and G near the tricritical point is given to the first correction as

$$\frac{h}{\phi^3} = x\left[\frac{G}{t^{\epsilon/2}}\left(\frac{x}{x+\frac{3}{2}}\right)^{\epsilon/2} + 1\right]^{-1/3} + \frac{1}{2}\left[\frac{G}{t^{\epsilon/2}}\left(\frac{x}{x+\frac{3}{2}}\right)^{\epsilon/2} + 1\right]^{-1} \tag{4.8}$$

where $x = t/\phi^2$.

It is interesting to be explicit about the energy density–energy density correlation function $\Gamma_{0,2}$ and the inverse susceptibility $\Gamma_{2,0}$ in the same approximation. We obtain

$$\Gamma_{0,2} = \left(C - \frac{2}{\epsilon}\right) + \frac{2}{\epsilon}\left[1 + \frac{G}{t^{\epsilon/2}}\left(\frac{x}{x+\frac{3}{2}}\right)^{\epsilon/2}\right]^{1/3} \tag{4.9}$$

$$\Gamma_{2,0} = t\left\{\left[\frac{G}{t^{\epsilon/2}}\left(\frac{x}{x+\frac{3}{2}}\right)^{\epsilon/2} + 1\right]^{-1/3} + \frac{3}{2x}\left[\frac{G}{t^{\epsilon/2}}\left(\frac{x}{x+\frac{3}{2}}\right)^{\epsilon/2} + 1\right]\right\}. \tag{4.10}$$

In the $h = 0$ limit (i.e. $x = \infty$) the result for $\Gamma_{2,0}$ is in agreement with other derivations (Rudnick and Nelson 1976, Lawrie 1976, Bruce and Wallace 1976). We note that equations (4.9) and (4.10), although valid near the tricritical point, i.e. for small $G/t^{\epsilon/2}$, give a correct extrapolation to the critical behaviour in the large x limit.

The effective critical exponents associated with the crossover region between critical and tricritical behaviour are just the same as those calculated in the preceding section as far as the dependence on the reduced variables $G/t^{\epsilon/2}$ and $G/h^{\epsilon/3}$ is considered. Such variables indeed, vary from zero to infinity either for t varying from zero to infinity at fixed G (the crossover between small correlation length region and the critical point) or for $G/t^{\epsilon/2}$ varying from zero (tricritical point) to infinity (critical point).

5. Conclusion

As far as the extrapolation of the procedure to the three-dimensional case is concerned, we want to note that the expressions of the critical exponents appear as a partial summing up of the ϵ expansion and, if the integration procedure is carried on directly in the $\epsilon = 1$ case then the structure of the relations is also modified (de Pasquale *et al* 1976). It must be stressed that the validity of such an extrapolation cannot be controlled because a qualitative property like the differentiability cannot be checked consistently, in any finite region of the (u, v) plane.

As a general comment we want to emphasize that the predicted universal behaviour in the whole range between $u = 1$ and $u = 0$ must be considered as an unphysical feature due to the model that involves only a quartic interaction among the fluctuations. Such a solution, as far as the crossover between the critical and the small correlation length regions is concerned, might be of physical interest in order to describe deviations from the asymptotic behaviour, but as long as a decreasing correlation length is considered deviations from the universal behaviour are expected because of higher order interactions among fluctuations in the Ising systems and also from various kinds of anisotropies in real magnetic systems.

The same criticism applies to our description of the tricritical behaviour. In this case it is possible to neglect ψ^6 interaction for non-negative ψ^4 interaction, but it amounts to a disregard of non-universal corrections in the crossover region.

It would be of interest to develop the next step of our procedure in order to see whether it amounts to just an improvement of the linear model or to an introduction of modifications on the structure of the parametric representation. A generalization of the parametric formulation of scaling has been proposed (Green *et al* 1971) and it would be relevant to support it by a microscopic approach. Such an improved approach might be also effective to overcome those inadequacies of the linear model analysed for instance by Barmatz *et al* (1975).

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